Invariant Points and Lines

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1 Definitions

Matrix Transformation A linear transformation represented by a 2x2 matrix.

Invariant Point A point (coordinate) that is not changed by a matrix transformation.

Invariant Line A line (in the form $y = mx + c$), which is not effected by a matrix transformation.

Line of invariant points A line of points which are individually all invariant points.

2 Calculating matrix transformations

Matrix transformations are extremely simple and can be done in the following way:

1. Define the transformation matrix i.e.: $\boldsymbol{T} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (reflection in the line $y = -x$)

2. Multiply this by a point i.e.: $\vec{v_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

3. Multiply the two values $\boldsymbol{T} \times \vec{v_1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3 Finding invariant points

First of all, the origin $\begin{bmatrix}0\\0\end{bmatrix}$ is never effected by a linear transformation. Why?

1. Define the transformation: $\boldsymbol{T} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$
2. Define the vertex $\begin{bmatrix}0\\0\end{bmatrix}$
3. Multiply the matrix by the vertex: $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}a\times 0 + b\times 0\\c\times 0 + d\times 0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

To find a line of invariant poins from the matrix $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$ (reflection in the line $y = -x$)

1. Define the equation: $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$
2. Multiply it out to give: $\begin{bmatrix}-y\\-x\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$
3. Then expand that to the equations: $-y = x$ and $-x = y$
4. Rearrange this to give: $y = -x$ and $y = -x$.
5. Clearly these are both the same, therefore, all the points are not effected by the reflection in the line $y = -x$ (this makes logical sense)

4 Finding invariant lines

Invariant lines are different to "lines of invaraint points" because the individual points on the line can move, but the line itself remains the same!
The best way to work this out (in my opinion):

1. Define the matrix transformation in the variable $\boldsymbol{T}$:
   $\boldsymbol{T} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

2. Multiply the transformation by the general formula for lines $y = mx + c$ and $y = mx^{'} + c$.

3. $\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ mx + c \end{bmatrix} = \begin{bmatrix} x^{'} \\ mx^{'} + c \end{bmatrix}$

4. $-x + mx + c = x^{'}$
   $-4x + 3(mx + c) = mx^{'} + c$

5. Substitute the value $x^{'}$ into the second formula!

$$\begin{aligned} -4x + 3(mx + c) &= m(-x + mx + c) + c \\ -4x + 3mx + 3c &= -mx + m^2x + mc + c \\ 2c - mc &= -4mx + m^2x + 4x \\ c(2 - m) &= x(-4m + m^2 + 4) \\ c(2 - m) &= x(m^2 - 4m + 4) \\ c(2 - m) &= x(m - 2)^2 \\ x(m - 2)^2 - c(2 - m) &= 0 \\ x(m - 2)^2 + c(m - 2) &= 0 \end{aligned} \tag{1}$$

6. You can see, to solve this formula for m, $m = 2$

7. Rewrite the formula with the value $m = 2$

8. Therefore, we can write the following formula: $y = 2x + c$

9. Where c is any value!

10. Therefore there is an invariant line for all lines with the gradient 2