Matrix Rotations

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1 Using the angle between two vectors

First of all, we can define the 2x2 matrix inversion in the variable $\boldsymbol{R}$ as:

$$\text{Where } \theta \text{ is the anti-clockwise angle of rotation.} \tag{1}$$

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

The general form of vectors is:

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

Now let's take the identity matrix:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

Now, let's take the vector of the positive x-axis and the positive y-axis:

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5}$$

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6}$$

And rotate this by the matrix $\boldsymbol{R}$:

$$\vec{c} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{7}$$

$$\vec{d} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{8}$$

Using this function, we can give evidence of the dot product between two vectors.
The dot product is defined as such (where a and b are random variables):

$$\cos\theta = \frac{x_a \times x_b + y_a \times y_b}{|a| \times |b|}$$

This can be re-arranged to find θ (theta):

$$\theta = \cos^{-1}\left(\frac{x_a \times x_b + y_a \times y_b}{|\vec{a}| \times |\vec{b}|}\right)$$

Now let's find the rotation between $\vec{a}$ & $\vec{c}$ and $\vec{b}$ & $\vec{d}$:

$$\begin{aligned}\theta_1 = \cos^{-1}\left(\frac{1 \times \cos\theta + 0 \times \sin\theta}{\sqrt{1^2 + 0^2} \times \sqrt{(\cos\theta)^2 + (\sin\theta)^2}}\right) \\ = \cos^{-1}\left(\frac{\cos\theta}{\sqrt{1^2 + 0^2} \times \sqrt{\cos^2\theta + \sin^2\theta}}\right) \\ = \cos^{-1}\left(\frac{\cos\theta}{1 \times 1}\right) \\ = \cos^{-1}(\cos\theta) \\ = \theta\end{aligned} \tag{9}$$

$$\begin{aligned}\theta_2 = \cos^{-1}\left(\frac{0 \times -\sin\theta + 1 \times \cos\theta}{\sqrt{0^2 + 1^2} \times \sqrt{(-\sin\theta)^2 + (\cos\theta)^2}}\right) \\ = \cos^{-1}\left(\frac{\cos\theta}{\sqrt{0^2 + 1^2} \times \sqrt{\sin^2\theta + \cos^2\theta}}\right) \\ = \cos^{-1}\left(\frac{\cos\theta}{1 \times 1}\right) \\ = \cos^{-1}(\cos\theta) \\ = \theta\end{aligned} \tag{10}$$

Therefore, we have proved that the rotation matrix $\boldsymbol{R}$ rotates a vector by the angle θ.

Alongside this proof, we have also managed to prove another fact: Given any 2d matrix, one can instantly tell if it's a rotation matrix by the following test:

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{11}$$

$$\sqrt{a^2 + c^2} = 1 \tag{12}$$

$$\sqrt{b^2 + d^2} = 1 \tag{13}$$

Because:

$$\sqrt{cos^2\theta + sin^2\theta} = 1 \tag{14}$$

$$\sqrt{sin^2\theta + cos^2\theta} = 1 \tag{15}$$

2 Clockwise or anti-clockwise

Considering that a rotation is always going to be in the range $0 \leq \theta \leq 360$ and considering the 3 functions we use in the matrix $\boldsymbol{R}$ are $\sin\theta$, $\cos\theta$ and $-\sin\theta$, we can plot these functions onto a graph:

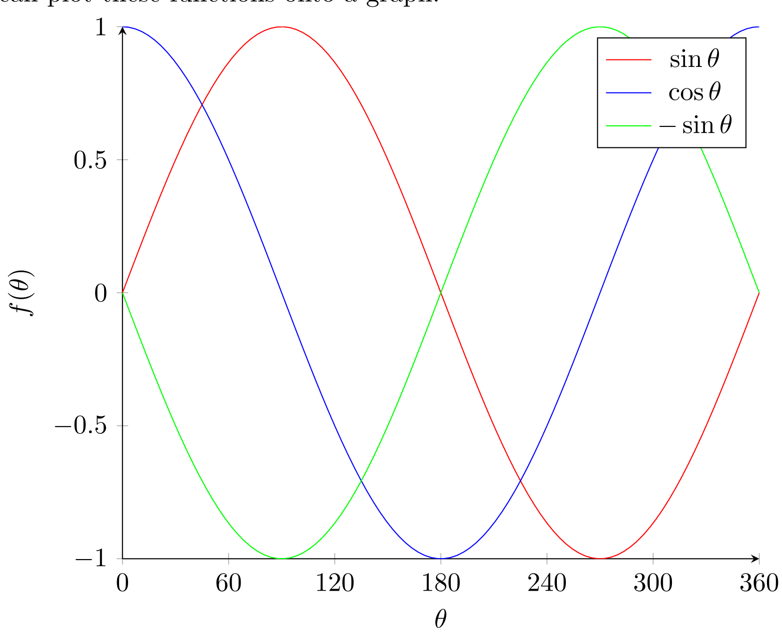

Figure 1: Domain: $0 \leq \theta \leq 360$, Range: $-1 \leq f(\theta) \leq 1$

Now, let's consider what an anti-clockwise and clockwise rotation actually is. Because we know that the matrix $\boldsymbol{R}$ gives an anti-clockwise rotation, we know that $0 \leq \theta \leq 180$ is anti-clockwise, hence $180 \leq \theta \leq 360$ is clockwise.
Let's zoom in to the $\cos\theta$ graph:

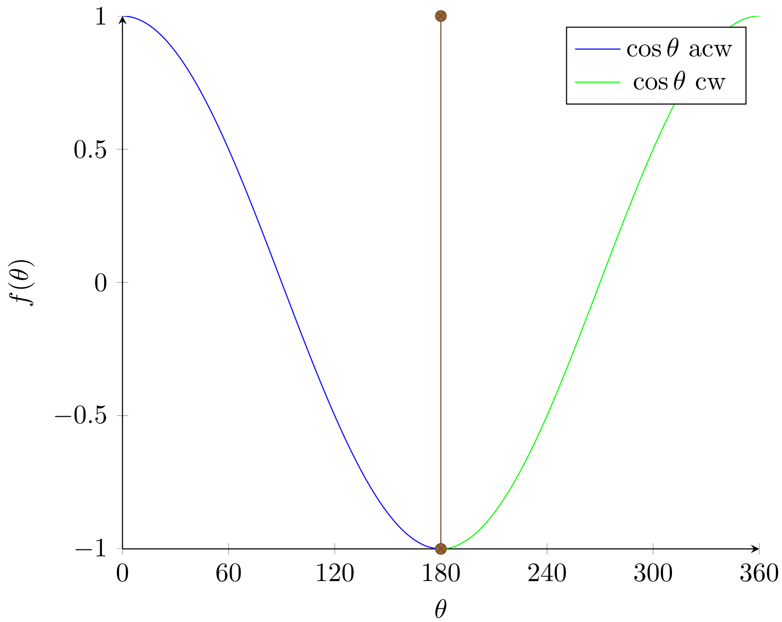

Figure 2: Domain: $0 \leq \theta \leq 360$, Range: $-1 \leq f(\theta) \leq 1$

Can you see how it doesn't matter whether the rotation is clockwise or anti-clockwise? It is negative or positive in either of these cases. This means we can forget about the cosine graph for now! Let's move onto $\sin\theta$ and $-\sin\theta$!

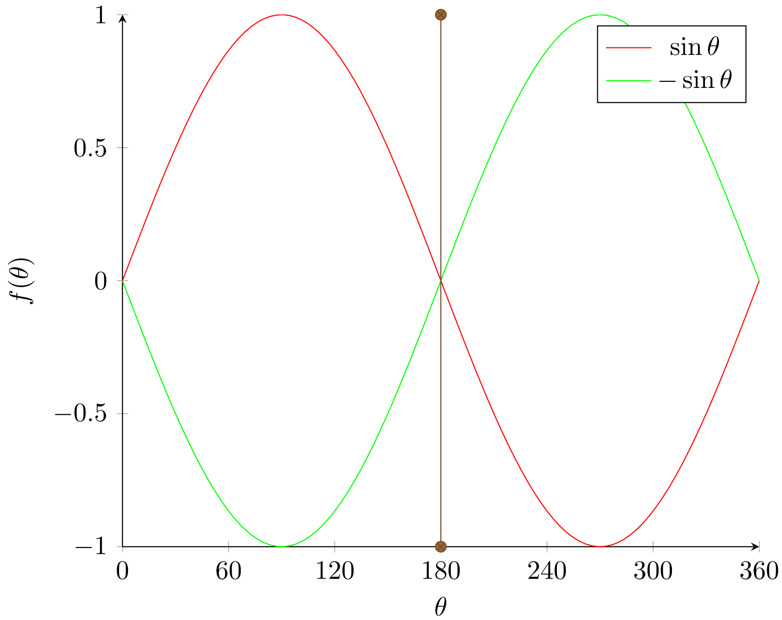

Figure 3: Domain: $0 \leq \theta \leq 360$, Range: $-1 \leq f(\theta) \leq 1$

Can you see how the graphs converge at 180^o? This means that in the domain $0 \leq \theta \leq 180$ (anti-clockwise), $0 \leq \sin\theta \leq 1$ and $-1 \leq -\sin\theta \leq 0$. Hence, for the domain $180 \leq \theta \leq 360$ (clockwise), $-1 \leq \sin\theta \leq 0$ and $0 \leq -\sin\theta \leq -1$.

But what does this actually mean for a mathematician? Well, it means, given the following rotation matrix:

$$\boldsymbol{B} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

We can see that $\boldsymbol{B}_{12}$ (the top right corner) is negative. Therefore, it is an anticlockwise rotation!

Now let's take another matrix $\boldsymbol{C}$:

$$\boldsymbol{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We can see that $\boldsymbol{C}_{21}$ (the bottom left corner) is negative. Therefore, it is a clockwise rotation!